MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS 1963 A

②

REPORT DOCUMENTATION PAGE

AD-A192 842

DTIC ELECTE S APR 0 4 1988 D

1b. RESTRICTIVE MARKINGS	DTIC FILE COPY

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
Technical Report No. 220	AFOSR·TR· 88 - 0 3 5 7

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
University of North Carolina		AFOSR/NM

6c. ADDRESS (City, State, and ZIP Code)	7b. ADDRESS (City, State, and ZIP Code)
Statistics Dept. 321-A Phillips Hall 039-A Chapel Hill, NC 27514	AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448

8a. NAME OF FUNDING / SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR	NM	AFOSR No. F49620 85C 0144.

8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS			
AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
	61102F	2304	A5	

11. TITLE (Include Security Classification)
Two classes of self-similar stable processes with stationary increments

12. PERSONAL AUTHOR(S)
Cambanis, S. and Maejima, M.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
preprint	FROM 9/87 TO 8/88	January 1988	32

16. SUPPLEMENTARY NOTATION
N/A

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Key Words & Phrases: Self-similar processes, stable processes, linear and harmonizable fractional processes, domain of attraction.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Two disjoint classes of self-similar symmetric stable processes with stationary increments are studied. The first class consists of linear fractional stable processes, which are related to moving average stable processes, and the second class consists of harmonizable fractional stable processes, which are connected to harmonizable stationary stable processes. The domain of attraction of the harmonizable fractional stable processes is also discussed.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	Unclassified/unlimited

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL
Maj. Brian Woodruff	(202) 767-5026	NM

DD FORM 1473, 84 MAR   83 APR edition may be used until exhausted.
All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified/Unlimited

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina
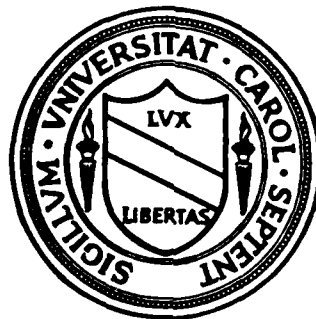
TWO CLASSES OF SELF-SIMILAR STABLE PROCESSES WITH STATIONARY INCREMENTS

by

Stamatis Cambanis

and

Makoto Maejima

88 3 31 029

TWO CLASSES OF SELF-SIMILAR STABLE PROCESSES WITH STATIONARY INCREMENTS[1]

Stamatis Cambanis

and

Makoto Maejima[2]

Center for Stochastic Processes
Statistics Department
University of North Carolina
Chapel Hill, NC 27599-3260 USA

Abstract Two disjoint classes of self-similar symmetric stable processes with stationary increments are studied. The first class consists of linear fractional stable processes, which are related to moving average stable processes, and the second class consists of harmonizable *fractional stable* processes, which are connected to harmonizable stationary stable processes. The domain of attraction of the harmonizable fractional stable processes is also discussed.

AMS 1980 Subject Classification: Primary 60G10, 60G99.

Key words and phrases: Self-similar processes, stable processes, linear and
    harmonizable fractional processes, domain of attraction.

1. Introduction

A stochastic process $X = (X(t))_{t \in \mathbb{R}}$ is self-similar with parameter $H \in \mathbb{R}$ (H-ss) if $X(c \cdot) \overset{d}{=} c^H X(\cdot)$ for all $c > 0$, and has stationary increments (si) if $X(\cdot + b) - X(b) \overset{d}{=} X(\cdot) - X(0)$ for all $b \in \mathbb{R}$, where $\overset{d}{=}$ means the equality of all finite dimensional distributions. A real-valued stochastic process $X = (X(t))_{t \in \mathbb{R}}$ is symmetric α-stable (SαS) if all linear combinations $\sum_{n=1}^{N} a_n X(t_n)$ have characteristic functions of the form $\exp\{-a|\theta|^\alpha\}$ for some $a > 0$. Here $\alpha \in (0,2]$ and when $\alpha = 2$, X is Gaussian.

In this paper, we study two disjoint classes of H-self-similar symmetric α-stable processes with stationary increments (H-ss si SαS processes). One consists of linear fractional stable processes, which are related to moving average stable processes, and the other consists of harmonizable fractional stable processes, which are connected to harmonizable stationary stable processes.

In Section 2 we give a representation of a SαS process with stationary increments in terms of a stationary SαS process which is either nonanticipating or fully anticipating. The representation is shown for $1 < \alpha \leq 2$, but it should in fact be valid for much larger classes than SαS processes.

In Section 3 we introduce the linear fractional stable processes $\Delta_{\alpha,H}(a,b) = (\Delta_{\alpha,H}(a,b;t))_{t \in \mathbb{R}}$ whose corresponding nonanticipating (or fully anticipating) stationary SαS process is a moving average. We show that when $1 < \alpha < 2$, to each line through the origin of the parameter plane (a,b) there corresponds a distinct linear fractional stable process (Theorem 3.1). This result implies that the two linear fractional processes defined independently in [12] and [21] are different.

In Section 4 we introduce the complex harmonizable fractional stable

processes $\theta_{\alpha,H}(a,b) = (\theta_{\alpha,H}(a,b;t))_{t\in\mathbb{R}}$, whose corresponding nonanticipating (or fully anticipating) stationary SαS process is harmonizable. When $1<\alpha\leq2$ we show that to each line through the origin of the parameter plane (a,b) there corresponds a distinct harmonizable fractional stable process (Theorem 4.1), and we study their domain of attraction in Section 6 (Theorems 6.1 and 6.2).

There is only one (distinct) real harmonizable fractional stable process, namely $\Re e\;\theta_{\alpha,H}(1,1)$, and it is shown in Section 5 that it is not a linear fractional stable process when $1<\alpha<2$ (Theorem 5.1).

The only reason the results in Sections 3-5 are stated only for $1<\alpha\leq2$ is that the representation in Section 2 is established only for $1<\alpha\leq2$. If the representation in Section 2 is proved also for $0<\alpha\leq1$, then the same proofs would establish the results in Sections 3-5 for all $0<\alpha<2$.

It should also be mentioned that, under mild regularity conditions, the linear and the harmonizable fractional stable processes introduced here seem to be the only self-similar, symmetric stable processes with stationary increments whose corresponding nonanticipating (or fully anticipating) stationary stable processes are moving averages and harmonizable respectively. This characterization result is still under study.

The authors gratefully acknowledge insightful discussions on self-similar stable processes with Murad Taqqu and Florin Avram.

2. Representation of si SαS processes

In this section we assume $1<\alpha\leq2$.

Let $X = (X(t))_{t\in\mathbb{R}}$ be a continuous in probability si SαS process with $1<\alpha\leq2$. Define

$$(2.1) \qquad Y(t) = \int_{-\infty}^{0} e^{u}[X(t)-X(t+u)]du = X(t) - \int_{-\infty}^{t} e^{-(t-u)}X(u)du, \qquad t\in\mathbb{R},$$

where the integrals exist a.s. Clearly $Y = (Y(t))_{t\in\mathbb{R}}$ is a continuous in probability, stationary SαS process which depends linearly on the past increments (or values) of X. We call Y the nonanticipating stationary SαS process corresponding to the si SαS process X, and a straightforward calculation shows that for all s$<$t,

$$(2.2) \qquad\qquad X(t)-X(s) = Y(t)-Y(s) - \int_s^t Y(v)dv.$$

One can also introduce the fully anticipating stationary SαS process Y corresponding to the si SαS process by

$$(2.3) \quad Y(t) = \int_0^\infty e^{-u}[X(t)-X(t+u)]du = X(t) - \int_t^\infty e^{-(v-t)}X(v)dv, \quad t\in\mathbb{R},$$

and derive likewise representation (2.2).

Of course the increments of a stationary process Y define a si process X via $X(t)-X(s) = Y(t)-Y(s)$, and so does the indefinite integral of a stationary process Y with a.s. locally integrable paths via $X(t)-X(s) = \int_s^t Y(v)dv$. In fact, a si process is the indefinite integral of a stationary process if and only if its paths are a.s. locally absolutely continuous; and is both the increment of some stationary process as well as the indefinite integral of some (other) stationary process if and only it its paths are a.s. locally absolutely continuous and its derivative process is the derivative of a stationary process. These two simple classes of si processes are distinct (with nonempty intersection) but their union is not broad enough to encompass all si processes (as is easily shown via examples). Representation (2.2) is therefore the only one generally available for all si processes.

Representation (2.2) follows also from Masani's representation of helixes in Banach spaces ([15]), by viewing X as a helix in $L_p(\Omega,\mathcal{F},P)$ for $1\leq p<\alpha$. It should be pointed out that (2.2) defines a si SαS process X for each stationary SαS process Y even when $0<\alpha\leq 1$, provided the integral in (2.2) is well defined;

for necessary and sufficient conditions see [6], Theorem 4.1. The conjecture here is that (2.2) would be valid for $0 < \alpha \leq 1$ as well, but there may be technical difficulties in its derivation. For the sake of completeness we indicate the simple proofs of (2.1) and (2.2) when $1 < \alpha \leq 2$.

<u>Proof of (2.1).</u> The integral in the middle expression in (2.1) will exist a.s. if

$$\int_{-\infty}^{0} e^{u} \, E|X(t)-X(t+u)| \, du < \infty.$$

(In fact this is a necessary and sufficient condition in this SαS case ([5]).) But the stationarity of the increments of X implies that $E|X(0)-X(u)|$ grows linearly in $|u|$. The equality of the two expressions a.s. is shown likewise. $\square$

<u>Proof of (2.2).</u> From (2.1) we obtain

$$\int_{s}^{t} Y(v)dv = \int_{s}^{t} X(v)dv - \int_{s}^{t} dv \int_{-\infty}^{v} e^{-(v-u)} X(u)du$$

$$= \int_{s}^{t} X(v)dv - \int_{-\infty}^{t} du X(u) \int_{s \vee u}^{t} e^{-(v-u)} dv$$

$$= \int_{s}^{t} X(v)dv - (e^{-s}-e^{-t}) \int_{-\infty}^{s} e^{u} X(u)du - \int_{s}^{t} (1-e^{u-t}) X(u)du$$

$$= -(e^{-s}-e^{-t}) \int_{-\infty}^{s} e^{u} X(u)du + \int_{s}^{t} e^{u-t} X(u)du$$

$$= \int_{-\infty}^{t} e^{-(t-u)} X(u)du - \int_{-\infty}^{s} e^{-(s-u)} X(u)du$$

$$= [Y(t) - X(t)] - [Y(x) - X(s)],$$

i.e. (2.2). The interchange of the order of integration is justified by Fubini's theorem as for (2.1). $\square$

The nonanticipating or fully anticipating stationary SαS process Y corresponding to the si SαS process X via the representation (2.2) is continuous in probability. Every continuous in probability stationary SαS process Y ($0 < \alpha \leq 2$) has a version of the form

$$(2.4) \qquad Y(t) = \int_{-\infty}^{\infty} a(t,u)dZ(u), \qquad t \in \mathbb{R},$$

where Z has independent SαS increments and control measure μ, i.e.,

$$(2.5) \qquad E \exp(ir\int f dZ) = \exp(-|r|^{\alpha}\int |f|^{\alpha}d\mu)$$

for $f \in L_{\alpha}(\mu)$. $(V(t))_{t \in \mathbb{R}}$ is a strongly continuous group of isometries in $L_{\alpha}(\mu)$ and $a(t,\cdot) = V(t)\{a(0,\cdot)\}$ ([8]). Since Y has a measurable modification (being continuous in probability), the kernel $a(t,u)$ has a version jointly measurable in (t,u) ([18]), and we obtain from (2.2) and (2.4), provided all integrals are well defined,

$$X(t) - X(s) = \int_{-\infty}^{\infty}[a(t,u) - a(s,u) - \int_{s}^{t}a(v,u)dv] \, dZ(u)$$

$$= \int_{-\infty}^{\infty}[b(t,u) - b(s,u)] \, dZ(u),$$

where

$$b(t,u) - b(s,u) = a(t,u) - a(s,u) - \int_{s}^{t}a(v,u)dv$$

$$= [V(t) - V(s) - \int_{s}^{t}V(v)dv]\{a(0,\cdot)\}(u).$$

The most important examples of stationary SαS processes are moving averages and harmonizable processes. Y is a SαS moving average process $(0<\alpha\leq2)$ if

$$Y(t) = \int_{-\infty}^{\infty}h(t-s)dM(s), \qquad t \in \mathbb{R},$$

where $M = (M(s))_{s \in \mathbb{R}}$ is a SαS motion (i.e., has stationary, independent SαS increments and Lebesgue control measure) and $h \in L_{\alpha} := L_{\alpha}(\text{Leb})$.

Up to this point we have only considered real-valued SαS processes. When dealing with harmonizable processes it is natural and convenient to consider complex-valued SαS processes. Indeed radially SαS complex random variables will suffice. A complex r.v. $X=X_1+iX_2$ is radially SαS if X_1 and X_2 are jointly SαS with radially symmetric distribution, i.e. with $z=z_1+iz_2$, $E \exp\{i\Re ezX\} =$

$E \exp\{i(z_1 X_1 + z_2 X_2)\} = \exp\{-c|z|^{\alpha}\}$ for some c>0. A complex-valued process $X = (X(t))_{t \in \mathbb{R}}$ is radially SαS if all complex linear combinations $\sum_{n=1}^{N} z_n X(t_n)$ are complex radially SαS r.v.'s. Y is a harmonizable stationary SαS process (0<$\alpha \leq$2) if

$$(2.6) \qquad Y(t) = \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \qquad t \in \mathbb{R},$$

where Z has complex, independent, radially SαS increments and finite spectral (control) measure μ, i.e.

$$(2.7) \qquad E \exp(i\Re e \bar{z}\int f dZ) = \exp(-|z|^{\alpha}\int |f|^{\alpha} d\mu)$$

for all complex numbers z and functions f in $L_{\alpha}(\mu)$ ([4]). (Unless the stable distribution of Z is radially symmetric, Y is not stationary ([23]).) When μ is absolutely continuous with respect to Lebesgue measure, $\varphi(\lambda) = d\mu(\lambda)/d\lambda$ is called the spectral density of Y. When μ is Lebesgue measure, Z is called a complex SαS motion and is denoted by $\tilde{M}$. A real-valued SαS process Y is harmonizable if

$$Y(t) = \Re e \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \qquad t \in \mathbb{R},$$

where Z is as above necessarily complex. For simplicity here we will consider complex harmonizable processes, while moving average processes will always be considered real.

3. The linear fractional stable processes

In this section we consider the linear fractional stable processes. Let 0<H<1, a,b$\in \mathbb{R}$, and M be a SαS motion defined in Section 2. For 0<$\alpha \leq$2, H$\neq$1/α, define

$$(3.1) \qquad \Delta_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \{a[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}]$$

$$+ b[(t-u)_-^{H-1/\alpha} - (-u)_-^{H-1/\alpha}]\} \, dM(u), \quad t\in\mathbb{R},$$

with the convention $0^\gamma = 0$ even for $\gamma < 0$; and for $1 < \alpha \leq 2$, $H = 1/\alpha$,

$$(3.2) \qquad \Delta_{\alpha,1/\alpha}(a,b;t) = a\Lambda(t) + bM(t), \quad t\in\mathbb{R},$$

where

$$(3.3) \qquad \Lambda(t) = \int_{-\infty}^{\infty} \{\log|t-u| - \log|u|\} \, dM(u), \quad t\in\mathbb{R}.$$

The process $\Delta_{\alpha,H}(1,0)$ was introduced in [21] and called fractional Lévy motion, and the process $\Delta_{\alpha,H}(1,1)$ was introduced in [12], and called fractional stable process. Λ in (3.3) is the log-fractional stable process defined in [11]. The process $\Delta_{\alpha,H}$, $H \neq 1/\alpha$, can be defined for any α, $0 < \alpha \leq 2$, but the process $\Lambda(t)$ can be defined only in the case $1 < \alpha \leq 2$. It is easy to see that the processes $(\Delta_{\alpha,H}(a,b;t))_{t\in\mathbb{R}}$ as in (3.1) and (3.2) are H-ss si SαS. In the rest of this section, we assume $1 < \alpha \leq 2$.

The nonanticipating stationary SαS process $Y_{\alpha,H}(a,b;\cdot)$ corresponding to the ss si SαS process $\Delta_{\alpha,H}(a,b;\cdot)$ of (3.1) is determined via (2.1) as follows: For $t\in\mathbb{R}$

$$Y_{\alpha,H}(a,b;t) = \int_{-\infty}^{0} e^u \left(\int_{-\infty}^{\infty} \{a[(t-v)_+^{H-1/\alpha} - (t+u-v)_+^{H-1/\alpha}]\right.$$

$$\left. + b[(t-v)_-^{H-1/\alpha} - (t+u-v)_-^{H-1/\alpha}]\} \, dM(v)\right) du$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{0} e^u \{a[(t-v)_+^{H-1/\alpha} - (t+u-v)_+^{H-1/\alpha}] + b[(t-v)_-^{H-1/\alpha} - (t+u-v)_-^{H-1/\alpha}]\} du\right) dM(v)$$

$$(3.4) \qquad = \int_{-\infty}^{\infty} h_{a,b}(t-v) \, dM(v),$$

where

$$h_{a,b}(x) = \int_{-\infty}^{0} e^u \{a[x_+^{H-1/\alpha} - (x+u)_+^{H-1/\alpha}] + b[x_-^{H-1/\alpha} - (x+u)_-^{H-1/\alpha}]\} \, du$$

$$(3.5) \qquad = ah_1(x) + bh_2(x).$$

$$(3.6) \qquad h_1(x) = x_+^{H-1/\alpha} - e^{-x}\int_0^{x_+} e^v v^{H-1/\alpha}dv.$$

$$(3.7) \qquad h_2(x) = x_-^{H-1/\alpha} - e^{-x}\int_{x_-}^{\infty} e^{-v}v^{H-1/\alpha}dv.$$

Hence $Y_{\alpha,H}(a,b)$ is a moving average of the SαS motion M, i.e. a linear process, and we call $\Delta_{\alpha,H}(a,b)$ a linear fractional stable process. $Y_{\alpha,H}(a,b)$ is a nonanticipating moving average of M only if $h_{a,b}(x) = 0$ for x<0, , and since for x<0, $h_1(x) = 0$ but $h_2(x) \neq 0$, this is true only when b=0.

As in (3.4), the nonanticipating stationary SαS process $Y_{\alpha,1/\alpha}(a,b)$ corresponding to the $(1/\alpha)$-ss si SαS process $\Delta_{\alpha,1/\alpha}(a,b;\cdot) = a\Lambda(\cdot) + bM(\cdot)$ is determined via (2.1) as the following moving average

$$(3.8) \qquad Y_{\alpha,1/\alpha}(a,b;t) = \int_{-\infty}^{\infty} g_{a,b}(t-u)dM(u),$$

where

$$(3.9) \qquad g_{a,b}(x) = ag_1(x) + bg_2(x),$$

$$g_1(x) = \log|x| - e^{-x}\int_{-\infty}^{x} e^v \log|v|dv,$$

$$g_2(x) = e^{-x} \quad \text{for x>0,} \quad = 0 \text{ for x<0.}$$

Therefore, $Y_{\alpha,1/\alpha}(a,b)$ is a nonanticipating moving average of M only when a=0.

The linear fractional stable processes $\Delta_{\alpha,H}(a,b)$ are indexed by the parameters $(a,b) \in \mathbb{R}^2$. When $\alpha=2$, the processes $\Delta_{2,H}(a,b)$ are Gaussian, called fractional Brownian motion, and by comparing their covariances we see that for all a and b with $|a|+|b| \neq 0$, the processes $\Delta_{2,H}(a,b)$ are multiples of the same Gaussian processes in distribution. The main purpose of this section is to show that if $1<\alpha<2$ each line through the origin of the parameter plane (a,b) determines a distinct process in distribution (up to scaling of course).

Theorem 3.1. Let $0<H<1$, $1<\alpha<2$, $|a|+|b|\neq0$, and $\Delta_{\alpha,H}(a,b;\cdot)$ be given by (3.1) and (3.2). Then we have

$$C_{\alpha,H}^{-1}(a,b) \; \Delta_{\alpha,H}(a,b;\cdot) \; \overset{d}{=} \; C_{\alpha,H}^{-1}(a',b') \; \Delta_{\alpha,H}(a',b';\cdot),$$

where $C_{\alpha,H}(a,b)$ is defined by (3.10) and (3.14) below, if and only if

(i) $a = a' = 0$ or

(ii) $b = b' = 0$ or

(iii) $aa'bb' \neq 0$ and $a/b = a'/b'$.

<u>Proof.</u> Recall that for two SαS random variables X and Y, by writing $E \exp\{irX\}$ $= \exp\{-|r|^{\alpha}||X||_{\alpha}^{\alpha}\}$, it follows that X and Y have the same distribution, $X \overset{d}{=} Y$, if and only if they have the same scale parameters, $||X||_{\alpha}^{\alpha} = ||Y||_{\alpha}^{\alpha}$. Thus to compare the marginal distributions of $\Delta_{\alpha,H}(a,b;t)$ and $\Delta_{\alpha,H}(a',b';t)$ we need to compute their scale parameters.

First assume $H \neq 1/\alpha$. We find from (3.1) and (2.5) that

$$||\Delta_{\alpha,H}(a,b;t)||_{\alpha}^{\alpha} = |t|^{\alpha H}\{ \; (|a|^{\alpha}+|b|^{\alpha}) \int_1^{\infty}||1-v|^{H-1/\alpha}-|v|^{H-1/\alpha}|^{\alpha}dv$$

$$+ \int_0^1 |a|1-v|^{H-1/\alpha} - b|v|^{H-1/\alpha}|^{\alpha}dv \; \}$$

(3.10) $$=: \; |t|^{\alpha H} C_{\alpha,H}^{\alpha}(a,b).$$

For simplicity we drop the subscripts of α,H. It follows that for each fixed $t \geq 0$,

$$C^{-1}(a,b)\Delta(a,b;t) \; \overset{d}{=} \; C^{-1}(a',b')\Delta(a',b';t),$$

where we are assuming of course that $|a|+|b| \neq 0 \neq |a'|+|b'|$.

In view of the one-to-one relationship between the si SαS process $\Delta(a,b;\cdot)$ of (3.1) and its corresponding nonanticipatory stationary SαS process $Y(a,b;\cdot)$ of (3.4),

(3.11) $$C^{-1}(a,b)\Delta(a,b;\cdot) \; \overset{d}{=} \; C^{-1}(a',b')\Delta(a',b';\cdot)$$

is equivalent to

$$C^{-1}(a,b)Y(a,b;\cdot) \overset{d}{=} C^{-1}(a',b')Y(a',b';\cdot),$$

which, in view of (3.4) and Kanter's theorem [9], is equivalent to

$$C^{-1}(a,b)h_{a,b}(t) = \epsilon\, C^{-1}(a',b')h_{a',b'}(t-\tau) \quad \text{a.e.}(t)$$

for some $\epsilon \in \{\pm 1\}$ and $\tau \in \mathbb{R}$ (depending on a,b,a',b'); and this in turn is equivalent to

$$C^{-1}(a,b)\{ah_1(t) + bh_2(t)\} = \epsilon\, C^{-1}(a',b')\{a'h_1(t-\tau) + b'h_2(t-\tau)\} \quad \text{a.e. }(t)$$

in view of (3.5), which is equivalent to

$$(3.12) \qquad \left\{ \frac{a}{C(a,b)} = \epsilon\, \frac{a'}{C(a',b')} \, , \quad \frac{b}{C(a,b)} = \epsilon\, \frac{b'}{C(a',b')} \right\}$$

To obtain the necessity of the last condition, take $t < \min(0,\tau)$, so that $h_1(t) = 0 = h_1(t-\tau)$ and differentiate $C^{-1}(a,b)bh_2(t) = \epsilon C^{-1}(a',b')b'h_2(t-\tau)$ to obtain

$$bC^{-1}(a,b)(-t)^{H-1/\alpha-1} = \epsilon\, b'C^{-1}(a',b')(\tau-t)^{H-1/\alpha-1} \quad \text{a.e. } t < \min(0,\tau)$$

from which $\tau = 0$ and $bC^{-1}(a,b) = \epsilon b'C^{-1}(a',b')$ follow, and thus also $aC^{-1}(a,b) = \epsilon a'C^{-1}(a',b')$.

If $a = 0$, by (3.12), $a' = 0$ and (3.12) is satisfied with $\epsilon = \text{sgn}(bb')$, since by (3.10)

$$C(0,b) = |b|\left\{ \int_1^\infty \left||1-v|^{H-1/\alpha} - |v|^{H-1/\alpha}\right|^\alpha dv + \int_0^1 |v|^{\alpha H-1} dv \right\}.$$

Hence if $a = a' = 0$ or if $b = b' = 0$, then (3.11) is satisfied.

Now assume $aa'bb' \neq 0$. Then (3.12) is equivalent to

$$(3.13) \qquad \left\{ \frac{a}{b} = \frac{a'}{b'} \, , \quad \frac{C^\alpha(a,b)}{|a|^\alpha} = \frac{C^\alpha(a',b')}{|a'|^\alpha} \, , \quad \frac{C^\alpha(a,b)}{|b|^\alpha} = \frac{C^\alpha(a',b')}{|b'|^\alpha} \right\}.$$

Putting

$$A = \int_1^\infty \left| |1-v|^{H-1/\alpha} - |v|^{H-1/\alpha} \right|^\alpha dv.$$

$$f(x) = \int_0^1 \left| |1-v|^{H-1/\alpha} - x|v|^{H-1/\alpha} \right|^\alpha dv.$$

$$g(x) = (1+|x|^\alpha)A + f(x),$$

we have from (3.10), $C^\alpha(a,b) = |a|^\alpha g(b/a) = |b|^\alpha g(a/b)$ and thus (3.13) is equivalent to

$$\left\{ \frac{a}{b} = \frac{a'}{b'}, \quad g(\frac{b}{a}) = g(\frac{b'}{a'}), \quad g(\frac{a}{b}) = g(\frac{a'}{b'}) \right\}$$

which is equivalent to $a/b = a'/b'$ since $g(1/x) = g(x)/|x|^\alpha$. This completes the proof when $H \neq 1/\alpha$.

When $H = 1/\alpha$, we find $||\Delta_{1/\alpha}(a,b;t)||_\alpha^\alpha = |t| \, C_{1/\alpha}^\alpha(a,b)$ with

$$(3.14) \qquad C_{\alpha,1/\alpha}^\alpha(a,b) = 2|a|^\alpha \int_1^\infty |\log(v-1) - \log v|^\alpha \, dv$$

$$+ \int_0^1 |a[\log(1-v) - \log v] + b|^\alpha \, dv$$

$$=: |a|^\alpha \{C + f(b/a)\},$$

where

$$f(x) = \int_0^1 |\log(1-v) - \log v + x|^\alpha \, dv.$$

The rest of the argument is similar to the case $H \neq 1/\alpha$, using instead expressions (3.8) and (3.9). $\qquad \Box$

The linear fractional stable processes have the following time domain symmetry

$$(3.15) \qquad \Delta_{\alpha,H}(a,b;\cdot) \overset{d}{=} \Delta_{\alpha,H}(a,b;-\cdot).$$

In general, if a self-similar process $X = (X(t))_{t \in \mathbb{R}}$ has a version of the form

$$X(t) = \int_{-\infty}^\infty [f(t-u) - f(-u)] \, dM(u), \quad t \in \mathbb{R},$$

where $M = (M(u))_{u \in \mathbb{R}}$ has stationary and symmetrically distributed increments, then X has the property (3.15). Indeed, by using the si property of M and the symmetry of the distribution of X, we have

$$X(\cdot) = \int_{-\infty}^{\infty} [f(\cdot-u) - f(-u)] \, dM(u)$$

$$\overset{d}{=} \int_{-\infty}^{\infty} [f(-v) - f(-\cdot-v) \, dM(v)$$

$$= -X(-\cdot) \overset{d}{=} X(-\cdot).$$

We now comment on the sample path properties of the linear fractional stable processes $\Lambda_{\alpha,H}(a,b;t)$ with $0<\alpha<2$. The fractional Brownian motion $\Lambda_{2,H}(a,b;t)$ has always a sample continuous version. Kolmogorov's moment criterion implies that if $X(t)$ is H-ss si with $0<H\leq 1$ and $E|X(t)|^P<\infty$ for some $p>1/H$, then $X(t)$ has a sample continuous version. Since $\Lambda_{\alpha,H}(a,b;t)$ is SαS, $E|\Lambda_{\alpha,H}(a,b;t)|^P<\infty$ for any $0<p<\alpha$. Thus, if $1/\alpha < H < 1$, then there exists p such that $1/H < p < \alpha$, and hence $\Lambda_{\alpha,H}(a,b;t)$ has a sample continuous version; still the paths have a.s. nowhere bounded variation, since H<1 ([24], Theorem 3.3). When $0 < H \leq 1/\alpha$, the kernels of the stable integrals defining $\Lambda_{\alpha,H}(a,b;t)$ have singular points, which implies that their sample paths are nowhere bounded (see [17] and [19], and for a special case [13]).

4. The harmonizable fractional stable processes

In this section, we introduce a new class of complex ss si SαS processes. Let $0 < \alpha \leq 2$, $0 < H < 1$, $a \geq 0$, $b \geq 0$, $a+b > 0$ and $\widetilde{M}$ be a complex SαS motion (introduced in Section 2). Define

$$(4.1) \qquad \theta_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \frac{e^{it\lambda}-1}{i\lambda} \left(a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha} \right) d\widetilde{M}(\lambda), \quad t \in \mathbb{R}.$$

It is easy to check that $\theta_{\alpha,H}(a,b;\cdot)$ are H-ss si SαS. When $\alpha=2$, $\Re e\ \theta_{2,H}(a,b;\cdot)$ is the same process as the fractional Brownian motion, which is the linear fractional stable process with $\alpha=2$. This can easily be checked by calculating the covariance functions of those processes. However, when $\alpha < 2$, $\theta_{\alpha,H}(a,b;\cdot)$ is a new class of H-ss si radially SαS processes, as will be shown in the next section for the case $1 < \alpha < 2$. In what follows, we assume $1 < \alpha \leq 2$ for technical reasons.

The nonanticipating stationary SαS process corresponding to $\theta_{\alpha,H}(a,b;\cdot)$ given by (2.1) is

$$(4.2) \quad Y_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} e^{it\lambda} \frac{1}{-i\lambda-1} (a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha})\ d\tilde{M}(\lambda), \quad t\in\mathbb{R},$$

and is thus harmonizable of the form (2.6) with

$$Z(\lambda) = \int_0^{\lambda} \frac{1}{-iu-1} (au_+^{1-H-1/\alpha} + bu_-^{1-H-1/\alpha})\ d\tilde{M}(u), \quad \lambda\in\mathbb{R}.$$

In view of the harmonizability of its nonanticipating stationary process we call the processes $\theta_{\alpha,H}$ harmonizable fractional stable processes. It is interesting to note that the fully anticipating stationary SαS process corresponding to $\theta_{\alpha,H}(a,b;\cdot)$ given by (2.3) is

$$(4.3) \quad \int_{-\infty}^{\infty} e^{it\lambda} \frac{1}{i\lambda-1} (a\lambda_+^{1-H-1/\alpha} + b\lambda_-^{1-H-1/\alpha})\ d\tilde{M}(\lambda), \quad t\in\mathbb{R}.$$

Both stationary processes in (4.2) and (4.3) are (H-1)-ss, harmonizable, and they are identically distributed.

The complex harmonizable fractional stable processes $\theta_{\alpha,H}(a,b;\cdot)$ of (4.1) are indexed by the parameters $(a,b) \in \mathbb{R}_+^2$. We now show that each ray through the origin of the parameter space $\mathbb{R}_+^2$ determines a distinct complex process in distribution (up to scaling of course). We also show that the real harmonizable fractional stable processes $\Re e\ \theta_{\alpha,H}(a,b;\cdot)$ are all multiples of each other in distribution, namely each $\Re e\ \theta_{\alpha,H}(a,b;\cdot)$ is a multiple of the process

(4.4) $\quad \Psi_{\alpha,H}(t) := \Re e \; \theta_{\alpha,H}(1,1;t) \;=\; \Re e \int_{-\infty}^{\infty} \frac{e^{it\lambda}-1}{i\lambda} \; |\lambda|^{1-H-1/\alpha} d\tilde{M}(\lambda), \qquad t\in\mathbb{R},$

$$= \int_{-\infty}^{\infty} \frac{\sin t\lambda}{\lambda} |\lambda|^{1-H-1/\alpha} dM_1(\lambda) - \int_{-\infty}^{\infty} \frac{\cos t\lambda - 1}{\lambda} |\lambda|^{1-H-1/\alpha} dM_2(\lambda),$$

where M_1 and M_2 are the real and imaginary parts of the complex SαS motion $\tilde{M}$. It follows by [16], Theorem 7.2, that M_1 and M_2 are real SαS motions, whose increments are independent at distinct points $\lambda_1 \ne \lambda_2$: $dM_1(\lambda_1) \perp\!\!\!\perp dM_2(\lambda_2)$, and are dependent at the same point λ in the following manner:

$$(dM_1(\lambda), \; dM_2(\lambda)) \;\overset{d}{=}\; 2^{\frac{1}{2}}(d\lambda)^{1/\alpha} R^{1/2}(G_1, \; G_2)$$

where R is positive $(\alpha/2)$-stable with $E \exp(-rR) = \exp(-r^{\alpha/2})$, $r \ge 0$, G_1 and G_2 are standard normal, and R, G_1, G_2 are independent. It should be mentioned here that $\theta_{\alpha,H}(1,1;t)$ is also mentioned in the recent paper [20].

<u>Theorem 4.1</u> Let $1<\alpha\le2$, $0<H<1$, $a\ge0$, $b\ge0$, $a+b>0$, and $\theta_{\alpha,H}(a,b;\cdot)$ be given by (4.1). Then we have

(4.5) $\qquad (a^{\alpha}+b^{\alpha})^{-1/\alpha} \theta_{\alpha,H}(a,b;\cdot) \;\overset{d}{=}\; (a'^{\alpha}+b'^{\alpha})^{-1/\alpha} \theta_{\alpha,H}(a',b';\cdot)$

if and only if

    (i) $a = a' = 0$ or

    (ii) $b = b' = 0$ or

    (iii) $aa'bb' \ne 0$ and $a/b = a'/b'$.

Also for all a,a',b,b' we have

(4.6) $\qquad (a^{\alpha}+b^{\alpha})^{-1/\alpha} \Re e \; \theta_{\alpha,H}(a,b;\cdot) \;\overset{d}{=}\; (a'^{\alpha}+b'^{\alpha})^{-1/\alpha} \Re e \; \theta_{\alpha,H}(a',b';\cdot).$

<u>Proof</u>. In view of (2.7) the complex radially SαS r.v.'s $\int fdZ$ and $\int gdZ$ have the same distribution if and only if their scale parameters $||\int fdZ||_{\alpha}^{\alpha} = \int|f|^{\alpha}d\mu$ and $||\int gdZ||_{\alpha}^{\alpha} = \int|g|^{\alpha}d\mu$ are equal. Thus to compare the marginal distributions of $\theta_{\alpha,H}(a,b;t)$ and $\theta_{\alpha,H}(a',b';t)$ we need to compute their scale parameter. For

simplicity we drop the subscripts α, H in θ and Y. We have

$$||\theta(a,b;t)||_{\alpha}^{\alpha} = \int_{-\infty}^{\infty} \left| \frac{e^{it\lambda}-1}{i\lambda} (a\lambda_{+}^{1-H-1/\alpha} + b\lambda_{-}^{1-H-1/\alpha}) \right|^{\alpha} d\lambda$$

$$= \int_{-\infty}^{\infty} \left| \frac{2}{\lambda} \sin(\frac{t\lambda}{2}) \right|^{\alpha} (a^{\alpha}\lambda_{+}^{\alpha(1-H)-1} + b^{\alpha}\lambda_{-}^{\alpha(1-H)-1}) \, d\lambda$$

$$= (a^{\alpha}+b^{\alpha}) \, 2^{\alpha} \int_{0}^{\infty} |\sin(\frac{t\lambda}{2})|^{\alpha} \, \lambda^{-(\alpha H+1)} \, d\lambda$$

$$= |t|^{\alpha H}(a^{\alpha}+b^{\alpha}) \, 2^{\alpha(1-H)} \int_{0}^{\infty} |\sin\lambda|^{\alpha} \, \lambda^{-(\alpha H + 1)} \, d\lambda$$

$$=: \, |t|^{\alpha H} \, (a^{\alpha}+b^{\alpha}) \, D_{\alpha,H}.$$

Hence it follows that for each fixed $t>0$,

$$(a^{\alpha}+b^{\alpha})^{-1/\alpha}\theta(a,b;t) \overset{d}{=} (a'^{\alpha}+b'^{\alpha})^{-1/\alpha}\theta(a',b';t).$$

In view of the one-to-one relationship between the si SαS process $\theta(a,b;\cdot)$ of (4.1) and its corresponding nonanticipating stationary SαS process $Y(a,b;\cdot)$ of (4.2), (4.5) is equivalent to

$$(a^{\alpha}+b^{\alpha})^{-1/\alpha}Y(a,b;\cdot) \overset{d}{=} (a'^{\alpha}+b'^{\alpha})^{-1/\alpha}Y(a',b';\cdot)$$

which, because of (2.7), is equivalent to

$$(4.8) \quad ||\sum_{n=1}^{N} \bar{z}_{n}(a^{\alpha}+b^{\alpha})^{-1/\alpha}Y(a,b;t_{n})||_{\alpha}^{\alpha} = ||\sum_{n=1}^{N} \bar{z}_{n}(a'^{\alpha}+b'^{\alpha})^{-1/\alpha}Y(a',b';t_{n})||_{\alpha}^{\alpha}.$$

But

$$(4.9) \quad ||\sum_{n=1}^{N} \bar{z}_{n}(a^{\alpha}+b^{\alpha})^{-1/\alpha}Y(a,b;t_{n})||_{\alpha}^{\alpha}$$

$$= (a^{\alpha}+b^{\alpha})^{-1} \int_{-\infty}^{\infty} |\sum_{n=1}^{N} \bar{z}_{n}e^{it_{n}\lambda}|^{\alpha} (a^{\alpha}\lambda_{+}^{\alpha(1-H)-1} + b^{\alpha}\lambda_{-}^{\alpha(1-H)-1}) \, (1+\lambda^{2})^{-\alpha/2} \, d\lambda$$

$$= (a^{\alpha}+b^{\alpha})^{-1} \int_{0}^{\infty} \{a^{\alpha}|\sum_{n=1}^{N} \bar{z}_{n}e^{it_{n}\lambda}|^{\alpha} + b^{\alpha}|\sum_{n=1}^{N} \bar{z}_{n}e^{-it_{n}\lambda}|^{\alpha}\} \, \lambda^{\alpha(1-H)-1}(1+\lambda^{2})^{-\alpha/2} \, d\lambda.$$

Hence if one of conditions (i), (ii), (iii) holds then (4.8) holds, and so does (4.5). This is the "if" part.

We next prove the "only if" part. Since (i) and (ii) are easy cases, we only show that (4.8) implies (iii). As usual, we consider (4.9), with N=2. Putting $c = b/a$ and taking $z_1=1$, $z_2=-i$, $t_1=0$, $t_2=t \geq 0$, (4.9) becomes

$$\frac{1}{1+c^\alpha} \int_0^\infty \{|1+ie^{it\lambda}|^\alpha + c^\alpha|1+ie^{-it\lambda}|^\alpha\} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda.$$

Therefore (4.8) becomes, with $c' = b'/a'$,

$$(4.10) \quad \frac{(c'^\alpha-c^\alpha)}{(1+c^\alpha)(1-c'^\alpha)} \int_0^\infty \{|1+ie^{it\lambda}|^\alpha - |1+ie^{-it\lambda}|^\alpha\} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2}d\lambda$$

$$= \frac{2^{\alpha/2}(c'^\alpha-c^\alpha)}{(1+c^\alpha)(1-c'^\alpha)} \int_0^\infty \{(1-\sin t\lambda)^{\alpha/2} - (1+\sin t\lambda))^{\alpha/2}\} \lambda^{\alpha(1-H)-1}(1+\lambda^2)^{-\alpha/2}d\lambda = 0$$

for all $t \geq 0$. We now show that (4.10) is possible only when $c = c'$. To this end, it is enough to show that

$$f(t) := \int_0^\infty \{(1+\sin t\lambda)^{\alpha/2} - (1-\sin t\lambda)^{\alpha/2}\} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda \neq 0$$

for some $t > 0$.

We first consider the case $1 \leq \alpha H < 2$. We have

$$f(t) = (\int_0^{\pi/2t} + \int_{\pi/2t}^\infty) \{(1+\sin t\lambda)^{\alpha/2}-(1-\sin t\lambda)^{\alpha/2}\} \lambda^{\alpha(1-H)-1} (1+\lambda^2)^{-\alpha/2} d\lambda$$

$$= \int_0^{\pi/2t} \{(1+\frac{2}{\pi}t\lambda)^{\alpha/2} - (1-\frac{2}{\pi}t\lambda)^{\alpha/2}\} \lambda^{\alpha(1-H)-1}(1+\lambda^2)^{-\alpha/2} d\lambda$$

$$- 2\int_{\pi/2t}^\infty \lambda^{\alpha(1-H)-1}(1+\lambda^2)^{-\alpha/2}d\lambda$$

$$> \frac{4}{\pi} t \int_0^{\pi/2t} \lambda^{\alpha(1-H)} (1+\lambda^2)^{-\alpha/2} d\lambda - 2 \int_{\pi/2t}^\infty \lambda^{-\alpha H-1} d\lambda.$$

In the following, $c_j(\cdot,\cdot)$ will denote positive constants depending only on the parameters in the parentheses. If $1 < \alpha H < 2$, for sufficiently small $t>0$ we have

$$f(t) > c_1(\alpha,H) \ t - c_2(\alpha,H) \ t^{\alpha H} > 0.$$

If $\alpha H = 1$, also for sufficiently small $t > 0$ we have

$$f(t) > c_3(\alpha) \ t \ \log\frac{1}{t} - c_4 \ t > 0.$$

We next consider the case $0 < \alpha H < 1$. By changing variables, we have

$$f(t) = t^{\alpha H} \int_0^\infty \{(1+\sin x)^{\alpha/2} - (1-\sin x)^{\alpha/2}\} \ x^{\alpha-\alpha H-1} \ (t^2+x^2)^{-\alpha/2} \ dx$$

$$= t^{\alpha H} \int_0^\infty \{(1+\sin x)^{\alpha/2} - (1-\sin x)^{\alpha/2}\} \ x^{-\alpha H-1} \ dx$$

$$+ t^{\alpha H} \int_0^\infty \{(1+\sin x)^{\alpha/2}-(1-\sin x)^{\alpha/2}\} \ \{(1+t^2/x^2)^{-\alpha/2}-1\} \ x^{-\alpha H-1} \ dx$$

$$(4.12) \quad =: \ t^{\alpha H} \{c_5(\alpha,H) + g(t)\}.$$

Since $x^{-\alpha H-1}$ is strictly decreasing on $(0,\infty)$, we have $c_5(\alpha,H) > 0$. Choosing $\epsilon > 0$ and $t > 0$ sufficiently small we obtain

$$|g(t)| \leq \int_0^\infty |(1+\sin x)^{\alpha/2}-(1-\sin x)^{\alpha/2}| \ \{1 - (1+t^2/x^2)^{-\alpha/2}\} \ x^{-\alpha H-1} \ dx$$

$$\leq 2 \int_0^\epsilon x^{-\alpha H}dx + 2 \int_\epsilon^\infty \frac{t^2}{x^2} x^{-\alpha H-1}dx$$

$$< \frac{2\epsilon^{1-\alpha H}}{1-\alpha H} + \frac{2t^2}{(\alpha H+2)\epsilon^{\alpha H+2}} \ < c_5(\alpha,H).$$

Hence by (4.12), we have $f(t) > 0$ for sufficiently small $t > 0$. This concludes the proof of the "only if" part.

The case of the real processes $\Re e \ \theta(a,b;\cdot)$ is easier to handle, because their scale parameter is as in (4.7), and in (4.8) we need to consider only real coefficients r_n, in which case (4.9) simplifies to

$$|| \sum_{n=1}^N r_n(a^\alpha+b^\alpha)^{-1/\alpha}\Re e \ Y(a,b;t_n)||_\alpha^\alpha = \int_0^\infty | \sum_{n=1}^N r_n e^{it_n\lambda}|^\alpha \ \lambda^{\alpha(1-H)-1} \ (1+\lambda^2)^{-\alpha/2} \ d\lambda$$

and shows that $(a^\alpha+b^\alpha)^{-1/\alpha}\Re e \ Y(a,b;\cdot) \overset{d}{=} (a'^\alpha+b'^\alpha)^{-1/\alpha}\Re e \ Y(a',b';\cdot)$ for all

a,b,a',b', and hence (4.6).   □

By a reasoning analogous to that at the end of Section 3, if $1/\alpha < H < 1$, then $\theta_{\alpha,H}(a,b;t)$ has a sample continuous version, whose paths have nowhere bounded variation. (When $0 < H \leq 1/\alpha$ the paths of $\theta_{\alpha,H}(a,b;t)$ are expected to be nowhere bounded.)

5. The harmonizable versus the linear fractional stable processes

In this section we show that, when $1<\alpha<2$, the real harmonizable fractional stable process introduced in Section 4 is different from the linear fractional stable processes discussed in Section 3.

Theorem 5.1 Let $1<\alpha<2$. Then the law of the real harmonizable fractional stable process $\Psi_{\alpha,H}(\cdot)$ of (4.4) is distinct from the laws of the linear fractional stable processes $\Delta_{\alpha,H}(a,b;\cdot)$ of (3.1) when $H \neq 1/\alpha$, and $\Lambda(\cdot)$ of (3.3) when $H=1/\alpha$.

Proof. We first consider the case $H \neq 1/\alpha$. Recall from (4.6) that $||\Psi_{\alpha,H}(t)||_\alpha$ $= |t|^H 2^{1/\alpha} D_{\alpha,H}^{1/\alpha} =: |t|^H \gamma_{\alpha,H}$ and from (3.10) that $||\Delta_{\alpha,H}(a,b;t)||_\alpha = |t|^H C_{\alpha,H}(a,b)$ $=: |t|^H \delta_{\alpha,H}(a,b)$. For simplicity we delete the subscripts α,H. We will assume

$$\gamma^{-1}\Psi(\cdot) \stackrel{d}{=} \delta^{-1}\Delta(a,b;\cdot)$$

and we will reach a contradiction. This implies the equality in distribution of the corresponding nonanticipating stationary SαS processes given in (4.2) and (3.4):

$$\left\{\gamma^{-1} \Re e \int_{-\infty}^{\infty} e^{it\lambda} \frac{|\lambda|^{1-H-1/\alpha}}{-i\lambda-1} d\tilde{M}(\lambda), \; t\in\mathbb{R}\right\} \stackrel{d}{=} \left\{\delta^{-1}\int_{-\infty}^{\infty} h_{a,b}(t-s)dM(s), \; t\in\mathbb{R}\right\}.$$

Introducing the independent radially SαS increments process N:

$$N(\lambda) = \frac{\delta}{\gamma} \int_0^\lambda \frac{|u|^{1-H-1/\alpha}}{(1+u^2)^{1/2}} \, d\widetilde{M}(u), \quad \lambda \in \mathbb{R},$$

with spectral measure υ: $d\upsilon(\lambda)/d\lambda = (\delta/\gamma)|\lambda|^{1-H-1/\alpha}(1+\lambda^2)^{-1/2}$, it follows that

$$\{\Re e \int_{-\infty}^\infty e^{it\lambda} dN(\lambda), \ t\in\mathbb{R}\} \ \overset{d}{=} \ \{\int_{-\infty}^\infty h_{a,b}(t-s) dM(s), \ t\in\mathbb{R}\}.$$

Since the spectral measure N has no atoms, the usual inversion relationship gives with $I=(x,y)$, $0\leq x<y$,

$$N_s(I) := \frac{1}{2} \{N(I) + \overline{N}(-I)\} = \frac{1}{2} \{N(y) - N(x) + \overline{N}(-x) - \overline{N}(-y)\}$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-iyt} - e^{-ixt}}{-it} \, \Re e[\int_{-\infty}^\infty e^{it\lambda} dN(\lambda)] \, dt,$$

where the limit is in probability or in $||\cdot||_\alpha$-norm. For simplicity we put

$$\frac{e^{-iyt} - e^{-ixt}}{-it} = \int_I e^{-iut} du =: f_I(t)$$

and drop the subscripts in $h_{a,b}$. It then follows that

$$N_s(I) \ \overset{d}{=} \ \lim_{t\to\infty} \frac{1}{2\pi} \int_{-T}^T f_I(t) \left\{ \int_{-\infty}^\infty h(t-s) dM(s) \right\} dt$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^\infty \left\{ \int_{-T}^T f_I(t)h(t-s) dt \right\} dM(s).$$

Now $||\int f dZ||_\alpha = ||f||_\alpha$ implies that $\int f_n dZ$ converges in $||\cdot||_\alpha$-norm if and only if f_n converges in L_α. It follows from the above convergence in $||\cdot||_\alpha$ norm as $T\to\infty$, that the integrand $\int_{-T}^T f_I(t)h(t-s) dt$ converges in L_α as $T\to\infty$, and hence for a.e.(s) along some subsequence $T_n \to \infty$. But

$$\lim_{T\to\infty} \int_{-T}^T f_I(t)h(t-s) dt = \int_{-\infty}^\infty f_I(t)h(t-s) dt \quad \text{for a.e.}(s)$$

by Lebesgue dominated convergence, since

$$\int_{-\infty}^\infty |f_I(t)h(t-s)| dt \leq \left[\int_{-\infty}^\infty \left|\frac{e^{-iyt} - e^{-ixt}}{-it}\right|^{\alpha'} dt\right]^{1/\alpha'} \left[\int_{-\infty}^\infty |h(t-s)|^\alpha dt\right]^{1/\alpha}$$

because $\alpha' > 1$ $(1/\alpha + 1/\alpha' = 1)$. We thus obtain

$$N_s(I) \stackrel{d}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_I(t)h(t-s)dt \right\} dM(s).$$

It follows likewise that the joint distribution of $N_s(I_1)$, $N_s(I_2)$ is the same as that of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_{I_1}(t)h(t-s)dt \right\} dM(s), \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_{I_2}(t)h(t-s)dt \right\} dM(s).$$

For disjoint intervals I_1 and I_2 in $\mathbb{R}_+$, the independence of the former pair of random variables implies that of the latter. But with $1 < \alpha < 2$, $\int g_1 dZ$ and $\int g_2 dZ$ are independent if and only if $g_1 g_2 = 0$ a.e. (in this complex case see [4]). Therefore with $I_1 = (0,x)$, $I_2 = (a,b)$, $0 < x < a < b$, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} h(t-s)dt \cdot \int_{-\infty}^{\infty} \frac{e^{-ibt} - e^{-iat}}{-it} h(t-s)dt = 0 \quad \text{a.e.}(s).$$

The same argument used to show integrability of $f_I(\cdot)h(\cdot-s)$, establishes the continuity of the integral $\int_{-\infty}^{\infty} f_I(t)h(t-s)ds$ as a function of s, since the map $\mathbb{R} \ni s \to h(\cdot-s) \in L_\alpha$ is continuous. Thus the above equality holds for all s. Putting $s=0$, we obtain

$$(5.1) \qquad \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} h(t)dt \cdot \int_{-\infty}^{\infty} \frac{e^{-i(b-a)} - 1}{-it} e^{-iat}h(t)dt = 0.$$

But for $h \in L_\alpha$ with Fourier transform $H \in L_{\alpha'}$, we have (cf. [22], Theorem 74)

$$H(x) = \frac{d}{dx} \int_{-\infty}^{\infty} h(t) \frac{e^{-ixt} - 1}{-it} dt \quad \text{a.e. } (x)$$

and

$$H(y+a) = \frac{d}{dy} \int_{-\infty}^{\infty} \frac{e^{-iyt}-1}{-it} e^{-iat}h(t)dt$$

(which follows by exactly the same proof as for Theorem 74 in [22]). Thus differentiating (5.1) with respect to x and $y=b-a$ we obtain $H(x)H(b) = 0$ for all $0 < x < b$, where H is the Fourier transform of h. It follows that $H = 0$ a.e. on $\mathbb{R}_+$;

and likewise on $\mathbb{R}_-$. Thus $H = 0$ which implies $h = 0$ a.e. Because of the linear independence of h_1 and h_2 in (3.6) and (3.7), $h_{a,b} = 0$ implies $a = 0 = b$ which contradicts $a + b \neq 0$.

The case $H = 1/\alpha$ can be treated similarly. □

Now consider for $1 < \alpha \leq 2$ the real harmonizable fractional stable process

$$\Psi_{\alpha,1/\alpha}(t) = \Re e \int_{-\infty}^{\infty} \frac{e^{it\lambda} - 1}{i\lambda} |\lambda|^{1-2/\alpha} \, d\widetilde{M}(\lambda), \quad t \in \mathbb{R},$$

which is $(1/\alpha)$-ss si $S\alpha S$ process. It is distinct from $S\alpha S$ motion (the simplest $(1/\alpha)$-ss si $S\alpha S$ process), as it does not have independent increments. As we have seen in Theorem 4.1 it is also distinct from the log-fractional stable process $\Lambda(\cdot)$, introduced in [11] as a new $(1/\alpha)$-ss in $S\alpha S$ process. Thus $\Psi_{\alpha,1/\alpha}(\cdot)$ is a new example of a $(1/\alpha)$-ss si $S\alpha S$ process.

In [7], Theorem 3.1, it was shown (using Beurling's theorem) that the class of complex nonanticipating, invertible, $S\alpha S$ moving average processes is disjoint from that of the regular, harmonizable, stationary $S\alpha S$ processes. The proof of Theorem 5.1 shows that the two classes of processes, $\{\Re e \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \ t \in \mathbb{R}\}$ and $\{\int_{-\infty}^{\infty} h(t-s) dM(s), \ t \in \mathbb{R}\}$, are disjoint. Namely, the entire class of real harmonizable stationary $S\alpha S$ processes is disjoint from that of real $S\alpha S$ moving average processes.

6. The domain of attraction of the harmonic fractional stable process

The domain of attraction of the linear fractional and of the log-fractional stable processes has been studied in [1], [2], [10], [11], [12]. In this section, we study the domain of attraction of the H-self-similar, symmetric α-stable process with stationary increments $\theta_{\alpha,H}(a,b;\cdot)$ introduced in Section 4. We start with $\theta_{\alpha,H}(1,1;\cdot)$.

Theorem 6.1. Let $1 < \alpha < 2$, $1-1/\alpha < H < 1$, $\tilde{M}$ be the complex SαS motion in (4.1), $\{Y_n\}_{n=-\infty}^{\infty}$ be the harmonizable SαS sequence

$$Y_n = \int_{-\pi}^{\pi} e^{in\lambda}\, d\tilde{M}(\lambda)$$

and

(6.1)
$$Z_n = \sum_{k=-\infty}^{\infty} c_{n-k} Y_k$$

with appropriate coefficients $\{c_n\}$. Let $\gamma = H+1/\alpha-1 \in (0,1)$. Define $\{c_n\}$ by $c_0 = 0$, $c_n = |n|^{\gamma-1}$ for $n \neq 0$. Then $\{Z_n\}$ is well-defined by (6.1) and as $n\to\infty$,

$$\frac{1}{n^H} \sum_{m=1}^{[nt]} Z_m \xrightarrow{d} C_\gamma \theta_{\alpha,H}(1,1;t), \qquad t > 0,$$

where $C_\gamma = 4\Gamma(\gamma)\cos(\gamma\pi/2)$, and $\xrightarrow{d}$ means the convergence of all finite dimensional distributions of the indicated process.

Proof. (Step 1) We show that $\{Z_n\}$ is well-defined. Since

$$\sum_{|k|\leq K} c_{n-k} Y_k = \int_{-\pi}^{\pi} e^{in\lambda} \left(\sum_{m=n-K}^{n+K} c_m e^{-im\lambda} \right) d\tilde{M}(\lambda),$$

it suffices to show that the Fourier series $\sum_{m=-\infty}^{\infty} c_m e^{-im\lambda}$ (i.e. the sequence of its partial sums) converges in $L_\alpha(-\pi,\pi)$. Since $\gamma < 1$, $c_m = |m|^{\gamma-1} \downarrow 0$ as $|m| \uparrow \infty$ and thus $\sum_{m=-\infty}^{\infty} c_m e^{-im\lambda}$ is convergent everywhere except at $\lambda \equiv 0 \pmod{2\pi}$ (see [3], pp. 87–88). Also since $\alpha > 1$, a necessary and sufficient condition for $\sum_{m=-\infty}^{\infty} c_m e^{-im\lambda} \in L_\alpha(-\pi,\pi)$ is $\sum_{m=-\infty}^{\infty} c_m^\alpha |m|^{\alpha-2} < \infty$ (see [3], p. 207), which is satisfied since $\alpha(\gamma-1)+\alpha-2 = \alpha(H-1)-1 < -1$. In fact the argument in [3], p. 208, shows that in this case the sequence of partial sums converges in $L_\alpha(-\pi,\pi)$. For completeness, we show this fact below.

For simplicity, we consider the convergence of

$$f_m(x) := \sum_{k=1}^{m} c_k \cos kx \qquad \text{to} \qquad f(x) := \sum_{k=1}^{\infty} c_k \cos kx$$

in $L_\alpha(0,\pi)$ as $m \to \infty$. By the same argument as in [3], p. 208, for $n > m$, if $\pi/(n-m+1) \leq x < \pi/(n-m)$,

$$\left| f_m(x) - f(x) \right| \leq \sum_{k=m+1}^{n} c_k + \left| \sum_{k=n+1}^{\infty} c_k \cos kx \right|$$

$$\leq \sum_{k=m+1}^{n} c_k + \frac{\pi}{x} c_n \leq \sum_{k=m+1}^{n} c_k + \text{const.}(n-m)c_n \leq 2 \sum_{k=m+1}^{n} c_k.$$

Hence, with $B_n = \sum_{k=m+1}^{n} c_k$,

$$\int_0^\pi \left| f_m(x)-f(x) \right|^\alpha dx = \sum_{n=m+1}^{\infty} \int_{\pi/(n-m+1)}^{\pi/(n-m)} \left| f_m(x)-f(x) \right|^\alpha dx$$

$$\leq \text{const.} \sum_{n=m+1}^{\infty} \int_{\pi/(n-m+1)}^{\pi/(n-m)} B_n^\alpha dx \leq \text{const.} \sum_{n=m+1}^{\infty} B_n^\alpha (n-m)^{-2}$$

$$\leq \text{const.} \sum_{\ell=1}^{\infty} B_{\ell+m}^\alpha \ell^{-2},$$

by a similar argument as in Theorem 2 in Appendix 22 of [3],

$$\leq o_m(1) + \text{const.} \sum_{\ell=1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} \leq \text{const.} \sum_{\ell=1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} \quad \text{for large } m.$$

We will show $\lim_{m\to\infty} \sum_{\ell=1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} = 0$ under our assumptions. Let

$$A := \sum_{\ell=1}^{\infty} c_\ell^\alpha \ell^{\alpha-2} < \infty.$$

For any $\epsilon > 0$, there exists $L = L(\epsilon)$ such that $\sum_{\ell=L+1}^{\infty} c_\ell^\alpha \ell^{\alpha-2} < \epsilon$. For any $m > LA^{1/(1-\gamma)\alpha}/\epsilon$,

$$\sum_{\ell=L+1}^{\infty} c_{\ell+m}^\alpha \ell^{\alpha-2} \leq \sum_{\ell=L+1}^{\infty} c_\ell^\alpha \ell^{\alpha-2} < \epsilon.$$

Note that

$$\frac{\ell}{\ell+m} \leq \frac{L}{m} < \frac{\epsilon}{A^{1/(1-\gamma)\alpha}} \quad \text{for } \ell \leq L.$$

Hence

$$c_{\ell+m} = \left[\frac{1}{\ell+m} \right]^{1-\gamma} < \frac{\epsilon^{1-\gamma}}{A^{1/\alpha}} c_\ell.$$

Therefore any $m > LA^{1/(1-\gamma)\alpha}/\epsilon$,

$$\sum_{\ell=1}^{L} c_{\ell+m}^{\alpha} \ell^{\alpha-2} < \frac{\epsilon^{(1-\gamma)\alpha}}{A} \sum_{\ell=1}^{L} c_{\ell}^{\alpha} \ell^{\alpha-2} \leq \epsilon^{(1-\gamma)\alpha}.$$

Hence

$$\limsup_{m\to\infty} \sum_{\ell=1}^{\infty} c_{\ell+m}^{\alpha} \ell^{\alpha-2} < \epsilon + \epsilon^{(1-\gamma)\alpha}.$$

and so

$$\lim_{m\to\infty} \sum_{\ell=1}^{\infty} c_{\ell+m}^{\alpha} \ell^{\alpha-2} = 0,$$

concluding

$$\lim_{m\to\infty} \int_0^\pi |f_m(x) - f(x)|^\alpha \, dx = 0.$$

(Step 2) We have for each $t > 0$,

$$Z_n(t) := \frac{1}{n^H} \sum_{m=1}^{[nt]} Z_m$$

$$= \frac{1}{n^H} \int_{-\pi}^{\pi} \left(\sum_{m=1}^{[nt]} e^{im\lambda} \right) \left(\sum_{k=-\infty}^{\infty} c_k e^{-ik\lambda} \right) d\tilde{M}(\lambda)$$

$$= \frac{1}{n^{\gamma-1/\alpha+1}} \int_{-n\pi}^{n\pi} \left(\sum_{m=1}^{[nt]} e^{imu/n} \right) \left(\sum_{k=-\infty}^{\infty} c_k e^{-iku/n} \right) d\tilde{M}\left(\frac{u}{n}\right)$$

$$\overset{d}{=} \int_{-n\pi}^{n\pi} \left(\frac{1}{n} \sum_{m=1}^{[nt]} e^{imu/n} \right) \left(\frac{1}{n^{\gamma}} \sum_{k=-\infty}^{\infty} c_k e^{-iku/n} \right) d\tilde{M}(u)$$

(6.2)
$$= \int_{-n\pi}^{n\pi} K_n(t,u) L_n(u) \, d\tilde{M}(u) =: \tilde{Z}_n(t).$$

where

$$K_n(t,u) = \frac{1}{n} \sum_{m=1}^{[nt]} e^{imu/n}, \qquad |u| \leq n\pi.$$

$$= \frac{1}{n} e^{iu/n} \frac{e^{iu[nt]/n} - 1}{e^{iu/n}}, \quad 0 < |u| \leq n\pi.$$

$$\xrightarrow[n\to\infty]{} \frac{e^{itu} - 1}{iu}, \qquad u \neq 0.$$

and

$$L_n(u) = \frac{1}{n^\gamma} \sum_{k=-\infty}^{\infty} c_k e^{-iku/n} , \qquad 0 < |u| \leq n\pi.$$

We now follow the approach of [14]. Let $f(\lambda)$ be a symmetric, twice differentiable function such that $f(\lambda) = 1$, $|\lambda| \leq \pi/4$; $f(\lambda) = 0$, $|\lambda| \geq \pi/2$; and $f(\lambda)$ is monotone decreasing for $\lambda > 0$. Then by [14], p. 138,

$$(6.3) \qquad c_k' := \frac{1}{2\pi} C_\gamma \int_{-\pi}^{\pi} e^{ik\lambda} |\lambda|^{-\gamma} f(\lambda) d\lambda = |k|^{\gamma-1} + O(|k|^{-2})$$

and

$$(6.4) \qquad \sum_{k=-\infty}^{\infty} c_k' e^{-ikx} = C_\gamma |x|^{-\gamma} f(x) \quad \text{for } |x| \leq \pi.$$
$$= C_\gamma |x|^{-\gamma} \quad \text{for } |x| \leq \pi/4,$$

except for $x = 0$. Now put

$$(6.5) \qquad \tilde{Z}_n'(t) = \int_{-n\pi}^{n\pi} K_n(t,u) L_n'(u) \, d\tilde{M}(u),$$

where

$$(6.6) \qquad L_n'(u) = \frac{1}{n^\gamma} \sum_{k=-\infty}^{\infty} c_k' e^{-iku/n}, \qquad 0 < |u| \leq n\pi,$$
$$= C_\gamma |u|^{-\gamma} f(\frac{u}{n})$$

by (6.4) and thus,

$$(6.7) \qquad L_n'(u) = C_\gamma |u|^{-\gamma}, \quad 0 < |u| \leq n\pi/4,$$

$$(6.8) \qquad L_n'(u) \leq C_\gamma |u|^{-\gamma}, \quad 0 < |u| \leq n\pi.$$

(Step 3) We show that for each $t > 0$ as $n \to \infty$,

$$(6.9) \qquad \tilde{Z}_n(t) - \tilde{Z}_n'(t) \to 0$$

in probability, or equivalently, in view of (6.2) and (6.5), that as $n \to \infty$

$$\int_{-n\pi}^{n\pi} |K_n(t,u)|^\alpha \, |L_n(u) - L_n'(u)|^\alpha \, du \; \to \; 0.$$

For $u \neq 0$ we have

$$|L_n(u) - L_n'(u)| \; = \; \frac{1}{n^\gamma} \, \Big| \sum_{k=-\infty}^{\infty} (c_k - c_k')e^{-iku/n} \Big| \; \leq \; \frac{const.}{n^\gamma} \; ,$$

since by (6.3), $c_k - c_k' = O(|k|^{-2})$. Also for $0 < |u| \leq n\pi$,

$$(6.10) \qquad |K_n(t,u)| = \frac{1}{n} \left| \frac{\sin\left[\frac{[nt]u}{2n}\right]}{\sin(\frac{u}{2n})} \right| = \frac{2}{|v|} \left| \sin(\frac{[nt]u}{2n}) \right| \, \left| \frac{\frac{u}{2n}}{\sin(\frac{u}{2n})} \right| \leq \frac{\pi}{|u|}.$$

Since $\gamma > 0$ and $\alpha > 1$, it follows that with $0 < a < n\pi$,

$$\int_{a<|u|<n\pi} |K_n(t,u)|^\alpha \, |L_n(u) - L_n'(u)|^\alpha \, du \; \leq \; \frac{const.}{n^{\alpha\gamma}} \int_{a<|u|<n\pi} \frac{1}{|u|^\alpha} \, du$$

$$(6.11) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \; \frac{const.}{n^{\alpha\gamma}} \; \longrightarrow \; 0 \quad \text{as } n \to \infty.$$

Thus it remains to show that

$$\int_{|u|<a} |K_n(t,u)|^\alpha \, |L_n(u) - L_n'(u)|^\alpha \, du \; \to \; 0 \qquad \text{as } n \to \infty.$$

However, by the argument in [14] (see Equations (2.10), (2.16), and the next one on p. 139 in [14]), we see that

$$\int_{|u|<n\pi} |K_n(t,u)|^2 \, |L_n(u) - L_n'(u)|^2 \, |u|^{q-1} \, du \; \to \; 0 \qquad \text{as } n \to \infty,$$

for any $q \in (0,1)$. Hence

$$\int_{|u|<a} |K_n(t,u)|^2 \, |L_n(u) - L_n'(u)|^2 \, |u|^{q-1} du \; \to \; 0 \qquad \text{as } n \to \infty,$$

which implies (6.11) by Hölder's inequality.

(Step 4) We finally show that for each $t > 0$,

$$(6.12) \qquad\qquad\qquad\qquad \tilde{Z}_n'(t) \; \to \; C_\gamma \theta_{,H}(1,1;t)$$

in probability, or equivalently that

$$(6.13) \quad ||\tilde{Z}_n^{\cdot}(t) - C_\gamma \theta_{\alpha,H}(1,1;t)||_\alpha^\alpha = \int_{-n\pi}^{n\pi} | K_n(t,u)L_n^{\cdot}(u) - C_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} |^\alpha du$$

$$+ \int_{|u| > n\pi} | C_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma} |^\alpha du$$

$$\to 0 \quad \text{as } n \to \infty.$$

The integral $\int_{|u| > n\pi}$ tends to 0 as $n \to \infty$ because the integrand in Lebesgue integrable over $\mathbb{R}$. For the integral over $|u| < n\pi$ we have from (6.6),

$$g_n(t,u) := K_n(t,u) L_n^{\cdot}(u) - C_\gamma \frac{e^{itu} - 1}{iu} |u|^{-\gamma}$$

$$= C_\gamma |u|^{-\gamma} \{ K_n(t,u) f(\frac{u}{n}) - \frac{e^{itu} - 1}{in} \}.$$

It then follows from (6.7) that as $n \to \infty$,

$$g_n(t,u) \to 0, \quad u \neq 0,$$

and from (6.8) and (6.10) that

$$|g_n(t,u)| \leq C_\gamma |u|^{-\gamma} \min \{1+t, \frac{2+\pi}{|u|} \}, \quad 0 < |u| \leq n\pi,$$

where the function on the right hand side is in $L_\alpha(\mathbb{R})$ since $\alpha\gamma = 1-\alpha(1-H) < 1$ and $\alpha(\gamma+1) = \alpha H+1 > 1$. Hence the dominated convergence theorem implies

$$\int_{-n\pi}^{n\pi} |g_n(t,u)|^\alpha du \to 0 \quad \text{as } n \to \infty$$

and thus (6.13) is established.

It then follows from (6.12) that as $n \to \infty$,

$$\sum_{j=1}^{J} a_j \tilde{Z}_n^{\cdot}(t_j) \xrightarrow{d} C_\gamma \sum_{j=1}^{J} a_j \theta_{\alpha,H}(1,1;t_j)$$

in probability, and the result follows from

$$\sum_{j=1}^{J} a_j Z_n(t_j) \overset{d}{=} \sum_{j=1}^{J} a_j \tilde{Z}_n(t_j) \ .$$

which is established as (6.2), and from (6.9).   □

The restriction $1-1/\alpha < H < 1$ in Theorem 6.1 was used only in the estimate (6.11). If the left hand side of (6.11) could be shown to converge to zero for $0 < H < 1$, then Theorem 6.1 would hold for all $0 < H < 1$.

With some modification of the proof of Theorem 6.1 along the idea of Major [14], we obtain the following generalization.

Theorem 6.2 Let $a \geq 0$, $b \geq 0$, $a+b > 0$ and define

$$A = \frac{1}{8\Gamma(\gamma)} \left\{ \frac{a+b}{\cos(\gamma\pi/2)} - i \frac{a-b}{\sin(\gamma\pi/2)} \right\},$$

where $\gamma = H+1/\alpha-1$ as in Theorem 6.1. In the assumption of Theorem 6.1, we replace $c_n = |n|^{\gamma-1}$, $n \neq 0$, by

$$c_n = \begin{cases} An^{\gamma-1}, & n > 0, \\ \bar{A}|n|^{\gamma-1}, & n < 0. \end{cases}$$

Then we have

$$\frac{1}{n^H} \sum_{m=1}^{[nt]} Z_m \overset{d}{\longrightarrow} \theta_{\alpha,H}(a,b;t).$$

Sketch of Proof. We give only the outling of the proof here and omit the details.

With the same function f, as in the paragraph preceding (6.3), we define, instead of the $\{c'_k\}$ of (6.3),

$$a'_k := \frac{1}{2\pi} K_\gamma \int_{-\pi}^{\pi} e^{ik\lambda} |\lambda|^{-\gamma} f(\lambda) d\lambda, \qquad b'_k := \frac{1}{2\pi} K'_\gamma \int_{-\pi}^{\pi} e^{ik\lambda} |\lambda|^{-\gamma} f(\lambda) \, \text{sgn} \, \lambda \, d\lambda,$$

where

$$K_\gamma = 2(A+\bar{A})\Gamma(\gamma)\cos(\gamma\pi/2), \qquad K'_\gamma = 2(A-\bar{A})\Gamma(\gamma)\sin(\gamma\pi/2).$$

Then by the same argument as in [14],

$$a_k' = \frac{1}{2}(A+\bar{A}) \; |k|^{\gamma-1} + O(k^{-2}), \qquad b_k' = \frac{1}{2}(A-\bar{A})|k|^{\gamma-1}\text{sgn } k + O(k^{-2})$$

implying

$$(6.14) \qquad c_k' := a_k' + b_k' = \begin{cases} A|k|^{\gamma-1} + O(k^{-2}), & k > 0, \\ \bar{A}|k|^{\gamma-1} + O(k^{-2}), & k < 0, \end{cases}$$

and also

$$\sum_{k=-\infty}^{\infty} a_k' \; e^{-ikx} = K_\gamma |x|^{-\gamma} f(x), \qquad \sum_{k=-\infty}^{\infty} b_k' \; e^{-ikx} = iK_\gamma' |x|^{-\gamma} f(x) \text{ sgn } x.$$

Now put

$$\tilde{Z}_n'(t) = \int_{-n\pi}^{n\pi} K_n(t,u) L_n'(u) \; d\tilde{M}(u),$$

with

$$L_n'(u) = \frac{1}{n^\gamma} \sum_{k=-\infty}^{\infty} c_k' \; e^{-iku/n}, \qquad 0 < u \le n\pi,$$

$$=: L_{n,1}'(u) + L_{n,2}'(u),$$

where

$$(6.15) \qquad L_{n,1}'(u) = K_\gamma |u|^{-\gamma} f(\tfrac{u}{n}), \qquad L_{n,2}'(u) = iK_\gamma' |u|^{-\gamma} f(\tfrac{u}{n}) \text{ sgn } u.$$

Note that from (6.14), $c_k - c_k' = O(k^{-2})$. So the same argument as in Step 2 of the proof of Theorem 6.1 concludes that as $n \to \infty$,

$$\int_{-n\pi}^{n\pi} |K_n(t,u)|^\alpha \; |L_n(u) - L_n'(u)|^\alpha \; du \; \to \; 0.$$

Step 4 will be handled as follows. First note that

$$K_\gamma = \frac{1}{2}(a+b), \qquad K_\gamma' = -\frac{1}{2}(a-b),$$

so that

$$a = K_\gamma + iK'_\gamma, \quad b = K_\gamma - iK'_\gamma.$$

Hence we have

$$\theta_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \frac{e^{itu}-1}{iu} \{(K_\gamma+iK'_\gamma)u_+^{-\gamma} + (K_\gamma-iK'_\gamma)u_-^{-\gamma}\} \, d\tilde{M}(u)$$

$$= \int_{-\infty}^{\infty} \frac{e^{itu}-1}{iu} \{K_\gamma|u|^{-\gamma} + iK'_\gamma|u|^{-\gamma}\mathrm{sgn}\ u\} \, d\tilde{M}(u).$$

and

$$||\tilde{Z}'_n(t) - \theta_{\alpha,H}(a,b;t)||_\alpha^\alpha = \int_{-n\pi}^{n\pi} | K_n(t,u)L'_n(u) - \frac{e^{itu}-1}{iu}|u|^{-\gamma}(K_\gamma+iK'_\gamma\mathrm{sgn}\ u)\ |^\alpha \, du$$

$$+ \int_{|u|>n\pi} | \frac{e^{itu}-1}{iu}|u|^{-\gamma}(K_\gamma+iK'_\gamma\mathrm{sgn}\ u)\ |^\alpha \, du.$$

The second integral above tends to zero. As to the first integral we have from (6.15) that the integrand without the α power is equal to

$$K_n(t,u)L'_{n,1}(u) - K_\gamma\frac{e^{itu}-1}{iu}|u|^{-\gamma} + K_n(t,u)L'_{n,2}(u) - iK_\gamma\frac{e^{itu}-1}{iu}|u|^{-\gamma}\mathrm{sgn}\ u$$

$$= K_\gamma|u|^{-\gamma}\{ K_n(t,u)f(\frac{u}{n}) - \frac{e^{itu}-1}{iu} \} + iK'_\gamma|u|^{-\gamma}\mathrm{sgn}\ u \{ K_n(t,u)f(\frac{u}{n}) - \frac{e^{itu}}{iu} \}.$$

By the same reasoning as in the proof of Theorem 6.1, the right hand side above tends to zero, and the integral itself also converges to zero. □

References

[1] Astrauskas, A. (1983): Limit theorems for linearly generated random variables. Lit. Math. J. $\underline{23}$, 122-134.

[2] Avram, F. and Taqqu, M.S. (1986): Weak convergence of moving averages with infinite variance. In: E. Eberlein and M.S. Taqqu, eds., Dependence in Probability and Statistics (Birkhauser, Boston), 399-415.

[3] Bary, N.K. (1964): A Treatise on Trigonometric Series, Vol. II (Pergamon, Oxford).

[4] Cambanis, S. (1982): Complex symmetric stable random variables and processes. In: P.K. Sen, ed., Contributions to Statistics: Essays in Honour of N.L. Johnson (North-Holland, New York), 63-79.

[5] Cambanis, S. and Miller, G. (1980): Some path properties of p-th order of symmetric stable processes. Ann. Probability $\underline{8}$, 1148-1156.

[6] Cambanis, S., Rosinski, J. and Woyczynski, W.A. (1985): Convergence of quadratic forms in p-stable random variables and θ_p-radonifying operators. Ann. Probability $\underline{13}$, 885-897.

[7] Cambanis, S. and Soltani, A.R. (1984): Prediction of stable processes: Spectral and moving average representations. Z. Wahrsch. verw. Geb. $\underline{66}$, 593-612.

[8] Hardin, C.D. (1982): On the spectral representation of symmetric stable processes. J. Multivariate Anal. $\underline{12}$, 385-401.

[9] Kanter, M. (1973): The L^p norm of sums of translates of a function. Trans. Amer. Math. Soc. $\underline{179}$, 35-47.

[10] Kasahara, Y. and Maejima, M. (1986): Weighted sums of i.i.d. random variables attracted to integrals of stable processes. Preprint.

[11] Kasahara, Y., Maejima, M. and Vervaat, W. (1987): Log-fractional stable processes, University of North Carolina Center for Stochastic Processes Technical Report No. 177.

[12] Maejima, M. (1983): On a class of self-similar processes. Z. Wahrsch. verw. Geb. $\underline{62}$, 235-245.

[13] Maejima, M. (1983): A self-similar process with nowhere bounded sample paths. Z. Wahrsch. verw. Geb. $\underline{65}$, 115-119.

[14] Major, P. (1981): Limit theorems for non-linear functionals of Gaussian sequences. Z. Wahrsch. verw. Geb. $\underline{57}$, 129-158.

[15] Masani, P. (1976): On helixes in Banach spaces. Sankhya Ser. A $\underline{38}$, 1-27.

[16] Masry, E. and Cambanis, S. (1984): Spectral density estimation for stationary stable processes. Stochastic Process. Appl. $\underline{18}$, 1-31.

[17] Rosinski, J. (1987): On path properties of certain infinitely divisible processes, University of North Carolina Center for Stochastic Processes Technical Report No. 207.

[18] Rosinski, J. and Woyczynski, W.A. (1986): On Itô stochastic integration with respect to p-stable motion: Inner clock, integrability of sample paths, double and multiple integrals. Ann. Probability 14, 271-286.

[19] Samorodnitsky, G. (1987): Maxima of symmetric stable processes, University of North Carolina Center for Stochastic Processes Technical Report No. 181.

[20] Taqqu, M.S. (1987): Random processes with long-range dependence and high variability. J. Geophys. Res. 92, No. D8, 9683-9686.

[21] Taqqu, M.S. and Wolpert, R.L. (1983): Infinite variance self-similar processes subordinate to Poisson measures. Z. Wahrsch. verw. Geb. 62, 53-72.

[22] Tischmarsh, E.C. (1967): Introduction to the Theory of Fourier Integrals (Clearndon Press, Oxford).

[23] Urbanik, K. (1968): Random measures and harmonizable sequences. Studia Math. 31, 61-88.

[24] Vervaat, W. (1985): Sample path properties of self-similar processes with stationary increments. Ann. Probability 13, 1-27.

END

DATE

FILMED

6-1988

DTIC